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Nonlinear evolution equations for two-dimensional surface waves in a fluid of finite depth

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Two-dimensional weakly nonlinear surface gravity–capillary waves in an ideal fluid of finite water depth are considered and nonlinear evolution equations which are correct up to the third order of wave steepness are derived including the applied pressure on the free surface. Since no assumptions are made on the length scales, the equations can be applied to a fluid of arbitrary depth and to disturbances with arbitrary wavelength. For one-dimensional gravity waves, these evolution equations are reduced to those derived by Matsuno (1992). Most of the known equations for surface waves are recovered from the new set of equations as special cases. It is shown that one set of equations has a Hamiltonian formulation and conserves mass, momentum and energy. The analysis for irrotational flow is extended to two-dimensional uniform shear flow.

1. Introduction

To describe the nonlinear wave phenomenon in the ocean, various evolution equations have been proposed and studied. In shallow water, the Boussinesq equations for two-dimensional waves, the Korteweg–de Vries (KdV) equation for uni-directional waves, the Kadomtsev–Petviashvili (KP) equation for weakly two-dimensional waves and their solitary wave solutions are very well-known and of interests to all disciplines. For a fluid of finite depth, the Stokes solution for periodic waves and the nonlinear Schrödinger equation for slowly varying wave envelopes have been extensively studied. For internal waves in a stratified fluid, other evolution equations in addition to the aforementioned ones also have been derived: for example, the intermediate long wave equation for a fluid of finite depth and the Benjamin–Ono equation for deep water (see Whitham 1974; Miles 1980; Mei 1989).

Since all these model equations have their own restrictions on the water depth, different evolution equations have to be used depending on the depth of interest for real applications. In other words, each model equation is valid only for waves of a specific wavelength. Therefore, an evolution equation valid for any water depth (or for any wavelength) is very valuable and desirable for many applications.

Recently, Matsuno (1992, 1993*a*) derived the nonlinear evolution equations for one-dimensional gravity waves in a fluid of arbitrary depth by using complex function theory and the set of equations analogous to the KP equation for weakly two-dimensional waves. He also studied similar sets of equations for gravity waves over a non-uniform bottom and for internal waves (Matsuno 1993*b*,*c*). Here the Matsuno

(1992) equations for one-dimensional waves, correct up to the second order of the wave steepness, are extended to two-dimensional third-order nonlinear surface waves by using the Fourier transform method, including the effects of pressure forcing on the free surface and surface tension. Although the effects of a non-uniform bottom could be taken into account, in particular, for slowly varying bottom topography, uniform water depth is assumed in this paper.

Nonlinear evolution equations derived in §§ 2 and 3 have a structure similar to the Boussinesq equation for long waves: the kinematic equation and the dynamic equation for the surface elevation and the horizontal velocity at free surface. But the new set of equations has a full linear dispersion relation for gravity–capillary waves with nonlinear corrections and has no restriction on wavelength. As an external forcing, the applied pressure on the free surface is considered. In §§ 4 and 5, we show that the equations can be reduced to the many known nonlinear models for surface waves mentioned earlier by imposing special assumptions on the wavelength or water depth. In § 6, it is shown that one set of equations for the velocity potential evaluated at the free surface is reduced to Hamilton's equations for surface waves found by Zakharov (1968) and conserves mass, momentum and energy. Finally, similar sets of equations are derived for two-dimensional uniform shear flow with a little modification in § 7.

2. Derivation of the dynamic equation

The governing equation for the velocity potential $\phi(\boldsymbol{x}, z, t)$ in an inviscid, incompressible and irrotational three-dimensional flow can be written, in Cartesian coordinates, as

$$\nabla^2\phi + \phi_{zz} = 0 \qquad \text{for} \quad -h_0 < z < \zeta(\boldsymbol{x}, t), \tag{2.1}$$

and the kinematic boundary conditions at free surface and uniform bottom are given by

$$\zeta_t + \boldsymbol{u}\cdot\nabla\zeta = w \qquad \text{at} \quad z = \zeta(\boldsymbol{x}, t), \tag{2.2}$$

$$w = 0 \qquad \text{at} \quad z = -h_0, \tag{2.3}$$

where $\boldsymbol{x} = (x, y)$, $\nabla = (\partial_x, \partial_y)$, $(\boldsymbol{u}, w) = (\nabla\phi, \phi_z)$ is the three-dimensional velocity vector, $\zeta(\boldsymbol{x}, t)$ is a displacement of the free surface and h_0 is a water depth.

The dynamic boundary condition at the free surface for pressure p is given by

$$p = P_0 + \gamma\nabla\cdot\boldsymbol{n} \qquad \text{at} \quad z = \zeta(\boldsymbol{x}, t), \tag{2.4}$$

where P_0 is the external pressure applied to the free surface, γ is the surface tension and $\boldsymbol{n}$ is the horizontal component of the outward normal vector defined by

$$\boldsymbol{n} = -\frac{\nabla\zeta}{\left[1 + (\nabla\zeta)^2\right]^{1/2}}. \tag{2.5}$$

With these governing equations and boundary conditions, the Boussinesq-like equations valid for arbitrary water depth will be derived in terms of the surface elevation and the horizontal velocity at the free surface.

First we can derive the dynamic equation by substituting $z = \zeta$ into the Euler equations:

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z = -\nabla p/\rho, \tag{2.6}$$

$$w_t + \boldsymbol{u}\cdot\nabla w + ww_z = -p_z/\rho - g, \tag{2.7}$$

where g is the gravitational acceleration. By using the chain rule for differentiation

$$\left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{z=\zeta} = \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} - \left.\boldsymbol{u}_z\right|_{z=\zeta}\left(\frac{\partial \zeta}{\partial t}\right), \tag{2.8}$$

$$\left.\frac{\partial \boldsymbol{u}}{\partial x_j}\right|_{z=\zeta} = \frac{\partial \tilde{\boldsymbol{u}}}{\partial x_j} - \left.\boldsymbol{u}_z\right|_{z=\zeta}\left(\frac{\partial \zeta}{\partial x_j}\right) \tag{2.9}$$

for a horizontal velocity vector at the free surface $\tilde{\boldsymbol{u}}$

$$\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v}) = \boldsymbol{u}(x, y, \zeta, t), \tag{2.10}$$

and similar expressions for w and p, the Euler equations (2.6)–(2.7) can be written in terms of all variables evaluated at the free surface, after imposing the boundary conditions at the free surface, as

$$\tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}} + g\nabla\zeta = -\nabla\left(P_0/\rho\right) - \left(\gamma/\rho\right)\nabla(\nabla\cdot\boldsymbol{n}) - (\mathrm{D}^2\zeta)\nabla\zeta, \tag{2.11}$$

where the vertical velocity at the free surface $\mathrm{D}\zeta$ is given, from (2.2), by

$$\mathrm{D}\zeta = \zeta_t + (\tilde{\boldsymbol{u}}\cdot\nabla)\zeta\,. \tag{2.12}$$

Notice that the dynamic equation (2.11) is exact for an inviscid and incompressible fluid of arbitrary depth and can be applied to both irrotational and rotational flows. For irrotational flow, the same equation can be derived directly from the Bernoulli equation

$$\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}{\phi_z}^2 + gz + p/\rho = C(t), \tag{2.13}$$

where $C(t)$ is an arbitrary function of time and can be set equal to zero. Substituting $z = \zeta$ into (2.13) and taking the two-dimensional gradient yields the dynamic equation (2.11) using the following condition of irrotational flow:

$$\tilde{v}_x - \tilde{u}_y = \zeta_x(\mathrm{D}\zeta)_y - \zeta_y(\mathrm{D}\zeta)_x\,. \tag{2.14}$$

From (2.13), we can also obtain the dynamic equation for the velocity potential at the free surface Φ:

$$\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + g\zeta = -\left(P_0/\rho\right) - \left(\gamma/\rho\right)(\nabla\cdot\boldsymbol{n}) + \frac{1}{2}\left[\frac{(\bar{\mathrm{D}}\zeta)^2}{1+(\nabla\zeta)^2}\right], \tag{2.15}$$

and, for $\boldsymbol{U} = \nabla\Phi$,

$$\boldsymbol{U}_t + \boldsymbol{U}\cdot\nabla\boldsymbol{U} + g\nabla\zeta = -\nabla\left(P_0/\rho\right) - \left(\gamma/\rho\right)\nabla(\nabla\cdot\boldsymbol{n}) + \tfrac{1}{2}\nabla\left[\frac{(\bar{\mathrm{D}}\zeta)^2}{1+(\nabla\zeta)^2}\right], \tag{2.16}$$

where

$$\Phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, \zeta, t), \qquad \boldsymbol{U} = \nabla\Phi = \tilde{\boldsymbol{u}} + (\mathrm{D}\zeta)\nabla\zeta, \tag{2.17}$$

$$\bar{\mathrm{D}}\zeta = \zeta_t + \nabla\Phi\cdot\nabla\zeta = \left[1+(\nabla\zeta)^2\right](\mathrm{D}\zeta). \tag{2.18}$$

In addition to this dynamic equation, the kinematic equation is required to close the system. In other words, one more relation between ζ and the velocity (or the velocity potential) at the free surface from kinematic considerations is necessary. Since all dynamic equations ((2.11), (2.15) and (2.16)) are exact for an ideal fluid, any of them can be chosen based on convenience in finding the kinematic equation for the same variable. In the following section, the kinematic equation for $\tilde{\boldsymbol{u}}$ will be pursued and one for Φ will be considered later.

3. Derivation of the kinematic equation

First we non-dimensionalize all physical variables as

$$(\boldsymbol{x}, z) = L(\boldsymbol{x}^*, z^*), \quad t = (L/U)t^*, \tag{3.1}$$

$$\phi = (UL)\phi^*, \quad (\boldsymbol{u}, w) = U(\boldsymbol{u}^*, w^*), \quad \zeta = L\,\zeta^*, \quad P_0 = (\rho U^2)P_0^*, \tag{3.2}$$

where L is a characteristic length in the problem, say the wavelength, and $U = (gL)^{1/2}$ is a characteristic speed.

In this problem, we introduce two parameters β and ϵ defined by

$$\beta = a/L, \qquad \epsilon = h_0/L. \tag{3.3}$$

The first parameter β represents the wave steepness with a characteristic wave amplitude a and, for weakly nonlinear waves, small β is assumed. Since the second parameter $\epsilon \to 0$ for shallow water and $\epsilon \to \infty$ for deep water, ϵ lies in $0 < \epsilon < \infty$ depending on the water depth. Hereafter, the asterisks for non-dimensional variables will be dropped.

Substituting the following expansions for ϕ and ζ into (2.1)–(2.3):

$$\phi(\boldsymbol{x}, z, t) = \beta\phi_1 + \beta^2\phi_2 + \beta^3\phi_3 + O(\beta^4), \tag{3.4}$$

$$\zeta(\boldsymbol{x}, t) = \beta\zeta_1 + \beta^2\zeta_2 + \beta^3\zeta_3 + O(\beta^4), \tag{3.5}$$

the governing equation and the boundary conditions can be written, at $O(\beta^n)$, as

$$\left(\nabla^2 + \partial_z^2\right)\phi_n = 0 \qquad \text{for} \quad -\epsilon < z < 0, \tag{3.6}$$

$$\phi_{nz} = f_n \qquad \text{at} \quad z = 0, \tag{3.7}$$

$$\phi_{nz} = 0 \qquad \text{at} \quad z = -\epsilon, \tag{3.8}$$

where

$$f_1 = \zeta_{1t}, \qquad f_2 = \zeta_{2t} + \nabla\cdot(\zeta_1\boldsymbol{u}_1^0), \tag{3.9}$$

$$f_3 = \zeta_{3t} + \nabla\cdot\left(\zeta_1\boldsymbol{u}_2^0 + \zeta_2\boldsymbol{u}_1^0 + \tfrac{1}{2}\zeta_1^2\nabla\zeta_{1t}\right), \tag{3.10}$$

and $\boldsymbol{u}_n^0 = \nabla\phi_n|_{z=0}$.

Also the horizontal velocity at the free surface can be expanded, in a Taylor series, about $z = 0$ as

$$\begin{aligned} \tilde{\boldsymbol{u}} &= \nabla\phi|_{z=\zeta} \\ &= \nabla\phi|_{z=0} + \zeta_1\nabla\phi_{1z}|_{z=0} + \zeta_2\nabla\phi_{1z}|_{z=0} + \zeta_1\nabla\phi_{2z}|_{z=0} + \tfrac{1}{2}\zeta_1^2\nabla\phi_{1zz}|_{z=0} + O(\beta^4) \\ &= \boldsymbol{u}^0 + \zeta\nabla\zeta_t + \zeta\nabla\Big[\nabla\cdot(\zeta\tilde{\boldsymbol{u}})\Big] - \tfrac{1}{2}\zeta^2\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) + O(\beta^4), \end{aligned} \tag{3.11}$$

where equation (3.7) for $\phi_{nz}|_{z=0}$ and $\boldsymbol{u}^0 = \tilde{\boldsymbol{u}} - \zeta\nabla\zeta_t + O(\beta^3)$ for nonlinear terms in (3.11) have been used. Once $\boldsymbol{u}^0$ can be represented in terms of $\tilde{\boldsymbol{u}}$ and ζ, the kinematic equation (3.11) with the dynamic equation (2.11) form the complete set of equations for $(\tilde{\boldsymbol{u}}, \zeta)$.

To find $\boldsymbol{u}^0$, it is necessary to solve (3.6) with (3.7)–(3.8) sucessively but, up to $O(\beta^3)$, it is equivalent to solving the following equations:

$$\left(\nabla^2 + \partial_z^2\right)\phi = 0 \qquad \text{for} \quad -\epsilon < z < 0, \tag{3.12}$$

$$\phi_z = \zeta_t + \nabla \cdot \left(\zeta\tilde{\boldsymbol{u}} - \tfrac{1}{2}\zeta^2\nabla\zeta_t\right) \equiv f(\boldsymbol{x},t) \qquad \text{at} \quad z = 0, \tag{3.13}$$

$$\phi_z = 0 \qquad \text{at} \quad z = -\epsilon. \tag{3.14}$$

By using the Fourier transform (FT) defined by

$$\hat{\phi}(\boldsymbol{k},z,t) = \mathscr{F}[\phi] \equiv \int_{-\infty}^{\infty} \phi(\boldsymbol{x},z,t)\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\ \mathrm{d}\boldsymbol{x}, \tag{3.15}$$

$$\phi(\boldsymbol{x},z,t) = \mathscr{F}^{-1}[\hat{\phi}] \equiv \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \hat{\phi}(\boldsymbol{k},z,t)\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\ \mathrm{d}\boldsymbol{k}, \tag{3.16}$$

where $\boldsymbol{k} = (k_1,k_2)$, the Neumann problem (3.12)–(3.14) can be solved.

After taking FT, the solution of (3.12)–(3.14) is found to be

$$\hat{\phi} = A(\boldsymbol{k},t)\cosh k(z+\epsilon), \qquad A(\boldsymbol{k},t) = \frac{\hat{f}(\boldsymbol{k},t)}{k\sinh k\epsilon}, \tag{3.17}$$

where $k^2 = k_1^2 + k_2^2$. Then the FT of $f(\boldsymbol{x},t)$ can be written as

$$\begin{aligned} \hat{f}(\boldsymbol{k},t) &= \mathscr{F}\left[\phi^0\right] k\ \tanh(k\epsilon) \\ &= -\mathscr{F}\left[\nabla\cdot\boldsymbol{u}^0\right]\tanh(k\epsilon)/k, \end{aligned} \tag{3.18}$$

and, after taking the inverse FT, we have

$$\begin{aligned} f(\boldsymbol{x},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\nabla\cdot\boldsymbol{u}^0)\ G(|\boldsymbol{x}-\boldsymbol{x}'|;\epsilon)\ \mathrm{d}\boldsymbol{x}' \\ &\equiv \boldsymbol{T}\cdot[\boldsymbol{u}^0], \end{aligned} \tag{3.19}$$

where

$$G(|\boldsymbol{x}|;\epsilon) = -\int_0^{\infty} \tanh(k\epsilon)\ J_0(k|\boldsymbol{x}|)\ \mathrm{d}k, \tag{3.20}$$

and $J_0(x)$ is the zeroth-order Bessel function. To obtain (3.19), the integral representation for the Bessel function J_0

$$J_0(k|x|) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k|x|\cos\theta}\ \mathrm{d}\theta, \tag{3.21}$$

and

$$\mathscr{F}^{-1}[\hat{f}(\boldsymbol{k},t)\hat{g}(\boldsymbol{k},t)] = \int_{-\infty}^{\infty} f(\boldsymbol{x}',t)g(\boldsymbol{x}-\boldsymbol{x}',t)\ \mathrm{d}\boldsymbol{x}', \tag{3.22}$$

have been used.

From (3.11), (3.13) and (3.19), the kinematic equation is given by

$$\zeta_t + \nabla\cdot\left(\zeta\tilde{\boldsymbol{u}} - \tfrac{1}{2}\zeta^2\nabla\zeta_t\right) - \boldsymbol{T}\cdot\left[\tilde{\boldsymbol{u}} - \zeta\nabla\zeta_t - \zeta\nabla\left(\nabla\cdot(\zeta\tilde{\boldsymbol{u}})\right) + \tfrac{1}{2}\zeta^2\nabla^2\tilde{\boldsymbol{u}}\right] = O(\beta^4), \tag{3.23}$$

where $\nabla(\nabla\cdot\tilde{\boldsymbol{u}}) = \nabla^2\tilde{\boldsymbol{u}} + O(\beta^2)$ from the condition of irrotational flow (2.14) has been used. By imposing the same approximation as for (2.11) on (3.23), the dynamic equation correct up to $O(\beta^3)$ can be written, in a non-dimensionalized form, as

$$\tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}} + \nabla\zeta = -\nabla P_0 - \left[\mathrm{D}\zeta_t + (\tilde{\boldsymbol{u}}\cdot\nabla\zeta)_t\right]\nabla\zeta + W\nabla\left[\nabla^2\zeta - \tfrac{1}{2}\nabla\cdot\left((\nabla\zeta)^2\nabla\zeta\right)\right] + O(\beta^4), \quad (3.24)$$

where $W = \gamma/(\rho g L^2)$. Equations (3.23) and (3.24) are the complete system, correct up to the third order in wave steepness, governing the evolution of ζ and $\tilde{\boldsymbol{u}}$.

By neglecting the third-order terms (or the cubic nonlinear terms) in (3.23) and (3.24), the set of evolution equations for two-dimensional waves, correct up to $O(\beta^2)$, can be obtained as

$$\zeta_t + \nabla\cdot(\zeta\tilde{\boldsymbol{u}}) - \boldsymbol{T}\cdot[\tilde{\boldsymbol{u}} - \zeta\nabla\zeta_t] = O(\beta^3), \quad (3.25)$$

$$\tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}} + \nabla\zeta = -\nabla P_0 - \zeta_{tt}\nabla\zeta + W\nabla(\nabla^2\zeta) + O(\beta^3), \quad (3.26)$$

which is the two-dimensional extension of the Matsuno (1992) equation with the external forcing and the surface tension (see equations (5.5) and (5.6) below). By substituting (3.25)–(3.26) into nonlinear terms having time derivatives in (3.23)–(3.24), we can obtain many other forms of evolution equations, all of which are asymptotically equivalent to (3.23)–(3.24).

4. Various limits

4.1. *Shallow water*

For shallow water, $\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}]$ has the following limit:

$$\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}] = -\epsilon(\nabla\cdot\tilde{\boldsymbol{u}}) - \tfrac{1}{3}\epsilon^3\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}) - \tfrac{2}{15}\epsilon^5\nabla^2\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}) + O(\epsilon^7) \quad \text{as} \quad \epsilon \to 0, \quad (4.1)$$

which can be found, from (3.19)–(3.20), by using

$$\int_0^\infty k^{2n+1}J_0(kr)\,\mathrm{d}k = 2\pi(-\nabla^2)^n\delta(\boldsymbol{x}), \quad (4.2)$$

where $\delta(\boldsymbol{x})$ is Dirac's delta function. Then equations (3.23)–(3.24) with $W = 0$ for long waves become, in a dimensional form,

$$\zeta_t + \nabla\cdot\left[(h_0+\zeta)\tilde{\boldsymbol{u}}\right] + \tfrac{1}{3}h_0^3\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}}) = -\tfrac{2}{15}h_0^5\nabla^2\left[\nabla^2(\nabla\cdot\tilde{\boldsymbol{u}})\right] + h_0\nabla\cdot(\zeta\nabla\zeta_t) + O(\alpha^3\epsilon^3, \alpha\epsilon^7), \quad (4.3)$$

$$\tilde{\boldsymbol{u}}_t + \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\boldsymbol{u}} + g\nabla\zeta = -\nabla\left(P_0/\rho\right) - \zeta_{tt}\nabla\zeta + O(\alpha^3\epsilon^3, \alpha\epsilon^7), \quad (4.4)$$

where $\alpha = a/h_0 = \beta/\epsilon$ and $\alpha = O(\epsilon^2)$ is assumed. This set of equations is the higher-order Boussinesq equation for two-dimensional waves with external forcing while the left-hand sides of (4.3)–(4.4) are the classical Boussinesq equation. Substituting the following relationship between $\tilde{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$:

$$\tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}} - \tfrac{1}{3}h_0^2\nabla^2\bar{\boldsymbol{u}} - \tfrac{2}{3}h_0\zeta\nabla^2\bar{\boldsymbol{u}} - \tfrac{1}{45}h_0^4\nabla^2(\nabla^2\bar{\boldsymbol{u}}) + O(\alpha^3\epsilon^2, \alpha\epsilon^6), \quad (4.5)$$

into (4.3) and (4.4) gives the usual form for the depth-mean horizontal velocity $\bar{\boldsymbol{u}}$, which is

$$\zeta_t + \nabla\cdot\left[(h_0+\zeta)\bar{\boldsymbol{u}}\right] = 0, \quad (4.6)$$

$$\bar{\boldsymbol{u}}_t + \bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}} + g\nabla\zeta = -\nabla\left(P_0/\rho\right) + \tfrac{1}{3}h_0^2\nabla^2\bar{\boldsymbol{u}}_t + \boldsymbol{C}, \quad (4.7)$$

where $\boldsymbol{C}$, a higher-order correction term to the classical Boussinesq equation, is given by

$$\boldsymbol{C} = \tfrac{1}{3}h_0^2\nabla\left(\bar{\boldsymbol{u}}\cdot\nabla Q - Q^2\right) + \tfrac{1}{3}h_0\left(2\zeta\nabla Q_t + 3Q_t\nabla\zeta\right) + \tfrac{1}{45}h_0^4\nabla(\nabla^2 Q_t), \quad Q = (\nabla\cdot\bar{\boldsymbol{u}}). \quad (4.8)$$

Equations (4.6) and (4.7) can be further reduced to the (forced) KdV equation for unidirectional waves and the KP equation for weakly two-dimensional waves (Whitham 1974; Wu 1987).

4.2. *Deep water*

In deep water, $\boldsymbol{T}\cdot[\boldsymbol{u}]$ can be reduced to

$$\begin{aligned} \boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}] &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\nabla\cdot\tilde{\boldsymbol{u}}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{x}' + O(\epsilon^{-2}) \\ &\equiv \boldsymbol{H}\cdot[\tilde{\boldsymbol{u}}] \qquad \text{as} \quad \epsilon\to\infty. \end{aligned} \quad (4.9)$$

Replacing $\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}]$ in (3.23) or (3.25) by $\boldsymbol{H}\cdot[\tilde{\boldsymbol{u}}]$ gives the evolution equations for two-dimensional waves in deep water.

When we take the Fourier transform of (3.23)–(3.24) with (4.9), by neglecting surface tension and external pressure and using

$$\mathscr{F}\left[\boldsymbol{H}\cdot\tilde{\boldsymbol{u}}\right] = \mathrm{i}(\boldsymbol{k}/|\boldsymbol{k}|)\cdot\mathscr{F}\left[\tilde{\boldsymbol{u}}\right], \qquad \mathscr{F}\left[\nabla\cdot\tilde{\boldsymbol{u}}\right] = -\mathrm{i}\boldsymbol{k}\cdot\mathscr{F}\left[\tilde{\boldsymbol{u}}\right], \quad (4.10)$$

$$\mathscr{F}\left[f(\boldsymbol{x},t)g(\boldsymbol{x},t)\right] = \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\hat{f}(\boldsymbol{k}-\boldsymbol{k}',t)\hat{g}(\boldsymbol{k}',t)\,\mathrm{d}\boldsymbol{k}', \quad (4.11)$$

we obtain the set of equations in the Fourier-space (or $\boldsymbol{k}$-space), which was first obtained by Phillips (1960) for the resonant interactions of gravity waves in deep water (equation (3.19) in his paper).

4.3. *Axisymmetric waves*

For axisymmetric waves, $\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}]$ can be simplified to

$$\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}] = \int_0^{\infty}(r'u^a)_{r'}G^a(r,r';\epsilon)\,\mathrm{d}r', \quad (4.12)$$

where $u^a = u^a(r,t)$ is a radial velocity, $r = |\boldsymbol{x}|$ and

$$G^a(r,r';\epsilon) = -\int_0^{\infty}\tanh(k\epsilon)J_0(kr')J_0(kr)\,\mathrm{d}k. \quad (4.13)$$

5. One-dimensional waves

For two-dimensional flow (or one-dimensional waves), $\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}]$ can be reduced, after performing the integration in (3.19) with respect to y, to $\mathscr{T}[\tilde{u}]$ defined by

$$\mathscr{T}[\tilde{u}] = -\frac{1}{2\epsilon}\mathscr{P}\int_{-\infty}^{\infty}\frac{\tilde{u}(x',t)}{\sinh\left[(\pi/2\epsilon)(x'-x)\right]}\,\mathrm{d}x', \quad (5.1)$$

which, for deep water, becomes

$$\mathscr{T}[\tilde{u}] = -\frac{1}{\pi}\mathscr{P}\int_{-\infty}^{\infty}\frac{\tilde{u}(x',t)}{x'-x}\mathrm{d}x' + O(\epsilon^{-2}) \equiv -\mathscr{H}[\tilde{u}] + O(\epsilon^{-2}) \qquad \text{as} \quad \epsilon\to\infty, \quad (5.2)$$

where $\mathscr{P}$ means the integration in a principal value sense and $\mathscr{H}[g]$ is the Hilbert transformation.

Then the governing equations for one-dimensional waves can be written, in a dimensional form, as

$$\zeta_t + \left(\zeta\tilde{u} - \tfrac{1}{2}\zeta^2\zeta_{xt}\right)_x - \mathscr{T}\left[\tilde{u} - \zeta\zeta_{xt} - \zeta\left(\zeta\tilde{u}\right)_{xx} + \tfrac{1}{2}\zeta^2\tilde{u}_{xx}\right] = O(\beta^4), \tag{5.3}$$

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + g\zeta_x = -\left(P_{0x}/\rho\right) - \zeta_{tt}\zeta_x - (\tilde{u}\zeta_x{}^2)_t + (\gamma/\rho)\left(\zeta_x - \tfrac{1}{2}\zeta_x{}^3\right)_{xx} + O(\beta^4), \tag{5.4}$$

where ϵ in (5.1) is replaced by h_0. By neglecting the cubic nonlinear terms with $P_0 = \gamma = 0$, (5.3)–(5.4) yield the second-order Matsuno (1992) equation

$$\zeta_t + (\zeta\tilde{u})_x - \mathscr{T}\left[\tilde{u} - \zeta\mathscr{T}\left[\tilde{u}_x\right]\right] = 0, \tag{5.5}$$

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + g\zeta_x = g\zeta_x\mathscr{T}\left[\zeta_x\right], \tag{5.6}$$

where $\zeta_t = \mathscr{T}[\tilde{u}] + O(\beta^2)$ and $\tilde{u}_t = -g\zeta_x + O(\beta^2)$ are substituted into (5.3)–(5.4) for the second-order terms.

5.1. *Uni-directional waves*

When the evolution equations (5.3)–(5.4) are linearized, we have

$$\zeta_t - \mathscr{T}[\tilde{u}] = 0, \qquad \tilde{u}_t + g\zeta_x = (\gamma/\rho)\,\zeta_{xxx}, \tag{5.7}$$

and, after taking the FT, system (5.7) can be written as

$$\hat{\zeta}_t - \mathrm{i}\tanh(kh_0)\hat{u} = 0, \qquad \hat{u}_t - \mathrm{i}gk\hat{\zeta} = \mathrm{i}(\gamma/\rho)k^3\hat{\zeta}. \tag{5.8}$$

By assuming

$$(\hat{\zeta}, \hat{u})(k,t) = (a(k), b(k))\exp(\mathrm{i}kc(k)t), \tag{5.9}$$

the linear dispersion relation for gravity–capillary waves can be found as

$$c^2 = \frac{g}{k}\left(1 + \frac{\gamma}{\rho g}k^2\right)\tanh(kh_0), \tag{5.10}$$

where $c(k)$ is a linear phase velocity. From $c(k) = c(-k)$ and, for real $\zeta(x,0)$,

$$a(-k) = a^*(k), \tag{5.11}$$

where $a^*(k)$ is the complex conjugate of $a(k)$, it can be shown that the positive (or negative) sign of $c(k)$ corresponds to the right-going (or left-going) waves. After choosing one of the signs for $c(k)$ and taking the inverse FT, we have the following linearized evolution equation for uni-directional waves:

$$\zeta_t + \int_{-\infty}^{\infty} K(x-\xi)\zeta_\xi(\xi,t)\,\mathrm{d}\xi = 0, \tag{5.12}$$

where

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\,\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k. \tag{5.13}$$

Equation (5.12) is the linear term of Whitham's equation (1974, §13.14) when the surface tension is neglected. As pointed out by Matsuno (1992), it would be interesting to find the nonlinear version of (5.12), but this has not been done yet.

5.2. *The nonlinear Schrödinger equation*

It is known that the slow modulation of the wave envelope is governed by the cubic Schrödinger equation. Since the evolution equations derived in this paper are correct up to the third-order wave slope, they can be reduced to the nonlinear Schrödinger equation in the appropriate limit.

First, assume that ζ and $\tilde{u}$ can be expanded as

$$\zeta(x,t,\xi,\tau) = \beta\zeta_1 + \beta^2\zeta_2 + \beta^3\zeta_3 + O(\beta^4), \tag{5.14}$$

$$\tilde{u}(x,t,\xi,\tau) = \beta\tilde{u}_1 + \beta^2\tilde{u}_2 + \beta^3\tilde{u}_3 + O(\beta^4), \tag{5.15}$$

where

$$\xi = \beta(x - C_g t), \qquad \tau = \beta^2 t, \tag{5.16}$$

and C_g is the group velocity to be determined. Also $\mathscr{T}[f(\xi,\tau)E^n]$ needs to be expanded as

$$\mathscr{T}\left[f(\xi,\tau)E^n\right] = \left(\mathscr{T}_0[f] + \beta\mathscr{T}_1[f] + \beta^2\mathscr{T}_2[f] + O(\beta^3)\right)E^n, \tag{5.17}$$

where $E^n = \exp(\mathrm{i}n(kx - \omega t))$ and, for $n \neq 0$,

$$\mathscr{T}_0[f] = -\mathrm{i}fT_H, \quad \mathscr{T}_1[f] = -\frac{f_\xi}{n}\frac{\partial T_H}{\partial k}, \quad \mathscr{T}_2[f] = \frac{\mathrm{i}f_{\xi\xi}}{2n^2}\frac{\partial^2 T_H}{\partial k^2}, \quad T_H = \tanh(nkh_0), \tag{5.18}$$

and, for $n = 0$,

$$\mathscr{T}_0[f] = 0, \quad \mathscr{T}_1[f] = -h_0 f_\xi, \quad \mathscr{T}_2[f] = 0. \tag{5.19}$$

After substituting all expansions into (5.3)–(5.4) with $\gamma = P_0 = 0$ and choosing the first-order solution as

$$(\zeta_1, \tilde{u}_1)(x,t,\xi,\tau) = (A(\xi,\tau), B(\xi,\tau))\left(\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}\right), \tag{5.20}$$

where c.c. stands for the complex conjugate, we can find the evolution equation for A at third order, which turns out to be the nonlinear Schrödinger equation

$$\mathrm{i}A_\tau + a_1 A_{\xi\xi} + a_2|A|^2 A = 0, \tag{5.21}$$

where

$$a_1 = \tfrac{1}{2}\omega''(k), \tag{5.22}$$

$$a_2 = -\frac{ck^3}{4\sinh^4 q}\left(\cosh 4q + 8 - 2\tanh^2 q\right) + \frac{2ck^3}{\sinh^2 2q}\frac{(C_g + 2c\cosh^2 q)^2}{gh_0 - C_g^2}, \tag{5.23}$$

$$\omega^2 = gk\tanh q, \quad c = \frac{\omega}{k}, \quad C_g = \frac{c}{2}\left(1 + \frac{2q}{\sinh 2q}\right) = \omega'(k), \quad q = kh_0. \tag{5.24}$$

Further, if ξ-dependence is neglected, the Stokes wave solutions can be obtained (Mei 1989, §12.2). For two-dimensional wave envelopes, we may derive the equation of Davey & Stewartson (1974) with similar approximations to (3.23) and (3.24).

5.3. *Weakly two-dimensional waves*

For weakly two-dimensional waves, (3.25) and (3.26) can be further reduced to equations derived by Matsuno (1993*a*) by assuming that all physical variables are slowly varying functions of y. Let the characteristic length in the y-direction L_y be large compared with the characteristic length in the x-direction L such that $L/L_y = \mu << 1$ and $\tilde{v}/\tilde{u} = \widetilde{\phi_y}/\widetilde{\phi_x} = O(\mu)$.

From (3.19), $\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}]$ can be written as

$$\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nabla\cdot\tilde{\boldsymbol{u}}(x', y'+y, t)\; G(R;\epsilon)\; \mathrm{d}x'\mathrm{d}y', \tag{5.25}$$

where G is given by (3.20) and

$$R^2 = (x'-x)^2 + y'^2. \tag{5.26}$$

For slow variable y (which can be scaled as μy), $\nabla\cdot\tilde{\boldsymbol{u}}$ can be expanded as

$$\nabla\cdot\tilde{\boldsymbol{u}}(x', y'+y, t) = \nabla\cdot\tilde{\boldsymbol{u}}(x', y, t) + y'\tilde{u}_{xy}(x', y, t) + y'^2\tilde{u}_{xyy}(x', y, t)/2 + O(\beta^3), \tag{5.27}$$

where $\beta = O(\mu^2)$ has been assumed. When we substitute (5.27) into (5.25) and differentiate (5.25) once with respect to x, it can be shown that

$$\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}_x] = \mathscr{T}\left[\tilde{u}_x + \tilde{v}_y\right] + \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y'^2\tilde{u}_{xxyy}(x', y, t)\; G(R;\epsilon)\; \mathrm{d}x'\mathrm{d}y' + O(\beta^3), \tag{5.28}$$

where (5.1) has been used for the first term in the right-hand side. On performing the integration with respect to y' and using the irrotational condition (2.14), we can obtain, after some manipulations, the expression for $\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}_x]$ as

$$\boldsymbol{T}\cdot[\tilde{\boldsymbol{u}}_x] = \mathscr{T}[\tilde{u}_x] + \tfrac{1}{2}\mathscr{T}[\tilde{v}_y] - \tfrac{1}{2}h_0\Big(1 + \mathscr{T}\mathscr{T}\Big)[\tilde{u}_{yy}] + O(\beta^3). \tag{5.29}$$

To obtain (5.29), the following representation of Dirac's delta function:

$$\delta(x) = \frac{1}{\pi}\int_0^{\infty}\cos(kx)\; \mathrm{d}k, \tag{5.30}$$

and the convolution integral given by (3.22) have been used.

From (3.25)–(3.26) and (5.29), the evolution equations for weakly two-dimensional waves can be obtained, in a dimensional form, as

$$\Big[\zeta_t + (\zeta\tilde{u})_x\Big]_x - \mathscr{T}\Big[\tilde{u} - \zeta\zeta_{xt}\Big]_x - \tfrac{1}{2}\mathscr{T}[\tilde{v}_y] + \tfrac{1}{2}h_0\Big(1 + \mathscr{T}\mathscr{T}\Big)[\tilde{u}_{yy}] = O(\beta^3), \tag{5.31}$$

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + g\zeta_x = -\left(P_{0x}/\rho\right) - \zeta_{tt}\zeta_x + \left(\gamma/\rho\right)\zeta_{xxx} + O(\beta^3), \tag{5.32}$$

where $\boldsymbol{T}\cdot[\zeta\nabla\zeta_t] = \mathscr{T}[\zeta\zeta_{xt}] + O(\beta^3)$ has been used. Since $\tilde{u} = O(\zeta) = O(\beta)$, $\tilde{v} = O(\mu\beta)$ and $\partial_y = O(\mu)$ are assumed with $\beta = O(\mu^2)$, (5.31)–(5.32) are correct up to second order in wave steepness β. To eliminate $\tilde{v}$ from (5.31), by differentiating (5.31) with respect to x and using $\tilde{v}_x = \tilde{u}_y + O(\beta^2)$, we can find the evolution equations derived by Matsuno (1993*a*) for $(\zeta, \tilde{u})$, with $P_0 = \gamma = 0$, as

$$\Big[\zeta_t + (\zeta\tilde{u})_x\Big]_{xx} - \mathscr{T}\Big[\tilde{u} - \zeta\mathscr{T}[\tilde{u}_x]\Big]_{xx} - \tfrac{1}{2}\mathscr{T}[\tilde{u}_{yy}] + \tfrac{1}{2}h_0\Big(1 + \mathscr{T}\mathscr{T}\Big)[\tilde{u}_{xyy}] = 0, \tag{5.33}$$

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + g\zeta_x = g\zeta_x\mathscr{T}\Big[\zeta_x\Big], \tag{5.34}$$

where $\zeta_t = \mathscr{T}[\tilde{u}] + O(\beta^2)$ and $\tilde{u}_t = -g\zeta_x + O(\beta^2)$ are substituted into (5.31)–(5.32) for the second-order terms having time derivatives, as before.

6. Hamiltonian formulation

By use of the relationship between $\tilde{\boldsymbol{u}}$ and $\boldsymbol{U} = \nabla\Phi$ given, from (2.17) and (2.18), by

$$\tilde{\boldsymbol{u}} = \boldsymbol{U} - (\zeta_t + \boldsymbol{U}\cdot\nabla\zeta)\nabla\zeta + O(\beta^4), \tag{6.1}$$

the kinematic equation (3.23) can be rewritten, in terms of ζ and Φ, as

$$\zeta_t + \nabla\cdot\left[\zeta\nabla\Phi - \tfrac{1}{2}\nabla(\zeta^2\zeta_t)\right] - \boldsymbol{T}\cdot\nabla\left[\Phi - \zeta\zeta_t - \tfrac{1}{2}\nabla\cdot(\zeta^2\nabla\Phi)\right] = O(\beta^4). \tag{6.2}$$

From (2.15) with the same order of approximation as that in (6.2), the dynamic equation for Φ is given by

$$\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2 + g\zeta = \\ -\left(P_0/\rho\right) + \tfrac{1}{2}\zeta_t{}^2 + \zeta_t\nabla\Phi\cdot\nabla\zeta + (\gamma/\rho)\left[\nabla^2\zeta - \tfrac{1}{2}\nabla\cdot\left((\nabla\zeta)^2\nabla\zeta\right)\right] + O(\beta^4). \tag{6.3}$$

For one-dimensional waves, (6.2) and (6.3) can be reduced to

$$\zeta_t + \left[\zeta\Phi_x - \tfrac{1}{2}(\zeta^2\zeta_t)_x\right]_x - \mathscr{T}\left[\Phi_x - (\zeta\zeta_t)_x - \tfrac{1}{2}(\zeta^2\Phi_x)_{xx}\right] = 0, \tag{6.4}$$

$$\Phi_t + \tfrac{1}{2}\Phi_x{}^2 + g\zeta = -\left(P_0/\rho\right) + \tfrac{1}{2}\zeta_t{}^2 + \zeta_t\zeta_x\Phi_x + (\gamma/\rho)\left(\zeta_x - \tfrac{1}{2}\zeta_x{}^3\right)_x. \tag{6.5}$$

Therefore equations (6.2)–(6.3) (or (6.4)–(6.5)) are the complete set of equations for ζ and Φ.

Zakharov (1968) has shown that the exact evolution equations for irrotational surface waves can be written in the form of Hamilton's equations in which the Hamiltonian is the total energy and the conjugate variables are ζ and Φ. With the total energy (divided by ρ) given by

$$\mathscr{E} = \frac{1}{2}\int \mathrm{d}x \int_{-h_0}^{\zeta} \mathrm{d}z\left[(\nabla\phi)^2 + \phi_z{}^2\right] + \int \mathrm{d}x\left[\tfrac{1}{2}g\zeta^2 + \tfrac{\gamma}{\rho}\left((1+(\nabla\zeta)^2)^{1/2} - 1\right)\right], \tag{6.6}$$

the canonical equations (Zakharov 1968; Miles 1977) can be written as

$$\zeta_t = \frac{\delta\mathscr{E}}{\delta\Phi}, \qquad \Phi_t = -\frac{\delta\mathscr{E}}{\delta\zeta}, \tag{6.7}$$

where δ represents the functional derivatives.

For simplicity, we consider here the case of one-dimensional waves without external forcing. To find the equations following from the Hamiltonian form, we first substitute the following expression into the nonlinear terms of (6.4) and (6.5):

$$\zeta_t = \mathscr{T}\left[\Phi_x - (\zeta\mathscr{T}\Phi_x)_x\right] - (\zeta\Phi_x)_x + O(\beta^3). \tag{6.8}$$

Then we have the evolution equations correct up to $O(\beta^3)$ as

$$\zeta_t + \left[\zeta\Phi_x - \tfrac{1}{2}(\zeta^2\mathscr{T}\Phi_x)_x\right]_x - \mathscr{T}\left[\Phi_x - (\zeta\mathscr{T}\Phi_x)_x + \left(\zeta\mathscr{T}(\zeta\mathscr{T}\Phi_x)_x\right)_x + \tfrac{1}{2}(\zeta^2\Phi_{xx})_x\right] = 0, \tag{6.9}$$

$$\Phi_t + \tfrac{1}{2}\Phi_x{}^2 + g\zeta = \tfrac{1}{2}(\mathscr{T}\Phi_x)^2 - \zeta\Phi_{xx}(\mathscr{T}\Phi_x) - (\mathscr{T}\Phi_x)\mathscr{T}(\zeta\mathscr{T}\Phi_x)_x + (\gamma/\rho)\left(\zeta_x - \tfrac{1}{2}\zeta_x{}^3\right)_x. \tag{6.10}$$

After using the Green theorem and the kinematic boundary condition (2.2), we have, from (6.6), the total energy $\mathscr{E}$ in the form

$$\mathscr{E} = \frac{1}{2}\int \Phi\zeta_t\,\mathrm{d}x + \frac{g}{2}\int \zeta^2\,\mathrm{d}x + \frac{\gamma}{\rho}\int\left[(1+\zeta_x{}^2)^{1/2} - 1\right]\,\mathrm{d}x. \tag{6.11}$$

Substituting (6.9) into (6.11) for ζ_t gives the required Hamiltonian (and the total energy) as

$$\mathscr{E} = \frac{1}{2}\int \mathrm{d}x\Big[\Phi(\mathscr{T}\Phi_x) + \zeta{\Phi_x}^2 - \zeta(\mathscr{T}\Phi_x)^2 + \zeta^2\Phi_{xx}(\mathscr{T}\Phi_x) + (\zeta\mathscr{T}\Phi_x)\mathscr{T}(\zeta\mathscr{T}\Phi_x)_x\Big]$$
$$+\frac{1}{2}\int \mathrm{d}x\left[g\zeta^2 + \frac{\gamma}{\rho}\left({\zeta_x}^2 - \tfrac{1}{4}{\zeta_x}^4\right)\right] + O(\beta^5), \qquad (6.12)$$

where integration by parts and the antisymmetry of $\mathscr{T}$,

$$\int_{-\infty}^{\infty} f(\mathscr{T}g)\,\mathrm{d}x = -\int_{-\infty}^{\infty} g(\mathscr{T}f)\,\mathrm{d}x, \qquad (6.13)$$

have been used and the range of integration $(-\infty,\infty)$ can be replaced with $(-\lambda/2,\lambda/2)$ for periodic waves of wavelength λ. With (6.12) and (6.13), it is easy to show that the canonical equations (6.7) give the evolution equations (6.9)–(6.10). An alternative way to derive equations (6.9) and (6.10) is to substitute the solution of the Laplace equation satisfying the kinematic boundary condition at the bottom into (6.6) and (6.7). Following this approach, Kuznetsov, Spector & Zakharov (1994) derived the second-order equations of (6.9)–(6.10) for deep water with $g=\gamma=P_0=0$.

In addition to the conservation of energy obvious from the Hamiltonian formulation, mass and momentum are also conserved quantities:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 0, \qquad m = \int \zeta\,\mathrm{d}x, \qquad (6.14)$$

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} = 0, \qquad \mathscr{M} = \int \zeta\Phi_x\,\mathrm{d}x, \qquad (6.15)$$

which can be proved from (6.9)–(6.10) by using (6.13) and noticing that

$$\int_{-\infty}^{\infty} \mathscr{T}f\,\mathrm{d}x = 0. \qquad (6.16)$$

7. Equations for two-dimensional uniform shear flow

When the vorticity is constant in the entire domain at the initial time, two-dimensional perturbations to this basic flow are irrotational owing to the conservation of vorticity. Therefore the analysis in the preceding sections based on the assumption of irrotational flow can also be applied to two-dimensional uniform shear flow with a little modification.

The dynamic equations can be obtained from either (2.11) or the Bernoulli equation for constant vorticity given by

$$\phi_t + \tfrac{1}{2}{\phi_x}^2 + \tfrac{1}{2}{\phi_z}^2 + gz + p/\rho + \omega_0 z\phi_x - \omega_0\psi = 0, \qquad (7.1)$$

where ω_0 is the constant vorticity and ψ is a streamfunction for perturbation to uniform shear flow. Either method gives the dynamic equation, for $\tilde{u}$,

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + g\zeta_x + \omega_0(\zeta\tilde{u}_x + \mathrm{D}_0\zeta) = -\left(P_{0x}/\rho\right) - \left(\gamma/\rho\right)n_{xx} - (\mathrm{D}_0{}^2\zeta)\zeta_x, \qquad (7.2)$$

or, for $U=\Phi_x$,

$$U_t + UU_x + g\zeta_x + \omega_0(\zeta U_x + \bar{\mathrm{D}}_0\zeta) = -\left(P_{0x}/\rho\right) - \left(\gamma/\rho\right)n_{xx} + \frac{1}{2}\left(\frac{(\bar{\mathrm{D}}_0\zeta)^2}{1+{\zeta_x}^2}\right)_x, \qquad (7.3)$$

where the Cauchy–Riemann relation between ϕ and ψ,

$$\partial_x\left[\psi(x,\zeta,t)\right] = \Phi_x\zeta_x - \bar{\mathrm{D}}_0\zeta, \tag{7.4}$$

has been used in (7.3) and

$$\mathrm{D}_0\zeta = \mathrm{D}\zeta + \omega_0\zeta\zeta_x, \qquad \bar{\mathrm{D}}_0\zeta = \bar{\mathrm{D}}\zeta + \omega_0\zeta\zeta_x = \left(1+\zeta_x{}^2\right)\mathrm{D}_0\zeta. \tag{7.5}$$

By using the same method as before, the kinematic equation is found, for $\tilde{u}$, as

$$\zeta_t+\omega_0\zeta\zeta_x+\left[\zeta\tilde{u}-\tfrac{1}{2}\zeta^2\zeta_{xt}\right]_x-\mathscr{T}\left[\tilde{u}-\zeta\zeta_{xt}-\zeta\left(\zeta\tilde{u}\right)_{xx}+\tfrac{1}{2}\zeta^2\tilde{u}_{xx}-\tfrac{1}{2}\omega_0\zeta(\zeta^2)_{xx}\right]=0, \tag{7.6}$$

or, for U,

$$\zeta_t+\omega_0\zeta\zeta_x+\left[\zeta U-\tfrac{1}{2}(\zeta^2\zeta_t)_x\right]_x-\mathscr{T}\left[U-(\zeta\zeta_t)_x-\tfrac{1}{2}\left(\zeta^2 U\right)_{xx}-\tfrac{1}{3}\omega_0(\zeta^3)_{xx}\right]=0. \tag{7.7}$$

By applying the same order of approximation to the dynamic equation (7.2) (or (7.3)) as that in the kinematic equation (7.6) (or (7.7)), the complete set of equations for ζ and $\tilde{u}$ (or U) can be obtained.

For shallow water with basic shear flow, the solitary wave solution of the KdV equation has been studied by Benjamin (1962) and Freeman & Johnson (1970). The new set of equations for uniform shear flow considered here is the generalization of the previous works to a fluid of finite depth which can be reduced, in shallow water, to the Boussinesq equation for bi-directional waves and the KdV equation for uni-directional waves.

8. Discussion

We have derived the evolution equations for two-dimensional surface waves which are correct up to the third order in wave steepness. They have a structure similar to the Boussinesq equation for shallow water and can be regarded as the generalized Boussinesq equation for a fluid of finite depth. Since no restrictions on the wavelength or the water depth are imposed, these systems can serve as the governing equations for a general initial value problem to study the evolution of waves with arbitrary characteristic length or the interaction of waves of different wavelength. However, the numerical techniques capable of handling these sets of equations are yet to be found and that will be the subject of future publications. Although we have shown that some nonlinear evolution equations can be recovered from the new set of equations, other weakly nonlinear models can also be derived with appropriate limits. For external forcing, a floating body in addition to the applied pressure on the free surface considered here may be included in our analysis for a possible application to ship hydrodynamics. The effects of a non-uniform bottom can be taken into consideration for two-dimensional flow by using conformal mapping as shown in Matsuno (1993*b*). For a slowly varying bottom $h(x)$ with $h/L = O(1)$ and $h_x << 1$, our set of equations (5.3)–(5.4) (or (6.4)–(6.5)) are still valid if $\mathscr{T}[g]$ is replaced with $\mathscr{T}_h[g]$ (Matsuno 1993*b*) given by

$$\mathscr{T}_h[g] = -\frac{1}{2h(x)}\mathscr{P}\int_{-\infty}^{\infty} g(x',t)\Big/\sinh\left[\frac{\pi}{2}\int_x^{x'}\frac{\mathrm{d}x''}{h(x'')}\right]\mathrm{d}x'. \tag{8.1}$$

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